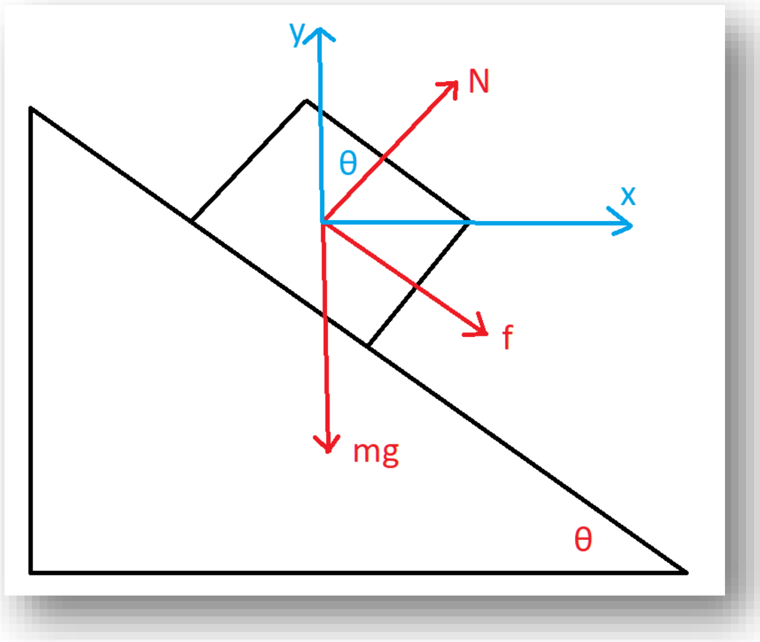
1. Problem 1

Let's look at a general free-body diagram of a car observed in a circular motion on a banked curve, moving in a circular motion:



The problem states that initially there is no friction, so that we can apply the second Newton's law and project the forces. In the x-direction, we sum the x-components as:

$$\sum F_x = ma_x$$

We only have the horizontal component of the normal force, which is:

$$\frac{N_x}{N} = \sin\theta$$
$$\therefore N_x = N\sin\theta$$

We will also consider the friction force in a general case (even though there is no friction for the given scenario, it will help us answer the question):

$$\frac{f_x}{f} = \cos\theta$$
$$\therefore f_x = f\cos\theta$$

So that:

$$N_x + f_x = ma_x$$

$$\therefore N\sin\theta + f\cos\theta = ma_x$$

The acceleration in the x-direction corresponds to the centripetal acceleration:

$$a_x = \frac{v^2}{r}$$

Hence:

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r}$$

Now if we project the forces in the y-direction:

$$\sum F_y = ma_y$$

Finding the components:

$$(mg)_y = -mg$$
$$\frac{f_y}{f} = -\sin\theta \quad \therefore \quad f_y = -f\sin\theta$$
$$\frac{N_y}{N} = \cos\theta \quad \therefore \quad N_y = N\cos\theta$$

We obtain:

$$N\cos\theta - f\sin\theta - mg = ma_y$$

No acceleration is present vertically, so that:

$$a_y = 0$$

Meaning:

$$N\cos\theta - f\sin\theta - mg = 0$$

We also know the definition of the friction force as:

$$f = \mu_k N$$

So that we have a set of 3 equations for any general case:

$$\begin{cases} N\sin\theta + f\cos\theta = \dfrac{mv^2}{r}, \\ N\cos\theta - f\sin\theta - mg = 0, \\ f = \mu_k N \end{cases}$$

Utilizing the definition of friction, we rewrite the first two equations as:

$$N\cos\theta - \mu_k N\sin\theta - mg = 0$$
$$N\sin\theta + \mu_k N\cos\theta = \frac{mv^2}{r}$$

Solving for the normal force using the first equation:

$$N(\cos\theta - \mu_k\sin\theta) = mg$$
$$N = \frac{mg}{\cos\theta - \mu_k\sin\theta}$$

Meaning:

$$N\sin\theta + \mu_k N\cos\theta = \frac{mv^2}{r}$$
$$\frac{mg}{\cos\theta - \mu_k\sin\theta}(\sin\theta + \mu_k\cos\theta) = \frac{mv^2}{r}$$

Here masses cancel out to yield:

$$\frac{v^2}{r} = \frac{g(\sin\theta + \mu_k\cos\theta)}{\cos\theta - \mu_k\sin\theta}$$

Solving for the velocity term:

$$v = \sqrt{\frac{gr(\sin\theta + \mu_k\cos\theta)}{\cos\theta - \mu_k\sin\theta}}$$

Dividing out the radical part by cosine:

$$v = \sqrt{\frac{gr(\tan\theta + \mu_k)}{1 - \mu_k\tan\theta}}$$

Initially, we know that there is no friction, so that:

$$v = v_o$$
$$\mu_k = 0$$

So that:

$$v_o = \sqrt{gr\tan\theta} \quad (1)$$

This formula holds as long as there is no linear acceleration and only centripetal acceleration is present. However, introducing the linear acceleration, the total acceleration of the object becomes:

$$a = \sqrt{a_t^2 + a_c^2}$$

An increase in velocity leads to an increase in the centripetal force, since:

$$a_c = \frac{v^2}{r}$$

First of all, according to the vector sum, the centripetal acceleration would **not** be pointing towards the center of the circular motion anymore, since we introduce a tangential acceleration vector. Since the net force is pointing along the acceleration vector, the direction of the net force would deviate from the center of the circular motion. In addition, its magnitude would increase, since there is an increase in the centripetal acceleration and the tangential acceleration is introduced. The magnitude and the direction of the normal force, as well as the force of gravity, would remain the same, since there is no force introduced along the vertical axis. On the other hand, to keep the car moving along the track, there must be a friction force of the tires introduced to our problem, so that now we can state that a coefficient of kinetic friction is involved in the problem. Otherwise, if no friction is present, and there is no force that balances an increase in the centripetal force, the car would skid. Looking at the derived formula:

$$v = \sqrt{\frac{gr(\tan\theta + \mu_k)}{1 - \mu_k\tan\theta}}$$

And we also know that:

$$v_o^2 = gr\tan\theta$$

Hence, we will express the variables in terms of the initial velocity:

$$v = \sqrt{\frac{gr\tan\theta + gr\mu_k}{1 - \mu_k \tan\theta}} = \sqrt{\frac{v_o^2 + \frac{v_o^2}{\tan\theta}\mu_k}{1 - \mu_k \tan\theta}} = \sqrt{\frac{v_o^2\left(1 + \frac{\mu_k}{\tan\theta}\right)}{1 - \mu_k \tan\theta}} = v_o\sqrt{\frac{1 + \frac{\mu_k}{\tan\theta}}{1 - \mu_k \tan\theta}}$$

We can see from here that the minimum value of velocity is the starting velocity when there is no friction involved and the velocity is constant:

$$\mu_k = 0 \quad \therefore \quad v = v_o$$

Now using some algebra, we will analyze the function $v(\mu_k)$ assuming that θ is constant, as it is given. Differentiating:

$$\frac{dv}{d\mu_k} = v_o \cdot \frac{1}{2\sqrt{\frac{1 + \frac{\mu_k}{\tan\theta}}{1 - \mu_k \tan\theta}}} \cdot \left(\frac{1 + \frac{\mu_k}{\tan\theta}}{1 - \mu_k \tan\theta}\right)'$$

We wish to find the critical points, such that:

$$\left(\frac{1 + \frac{\mu_k}{\tan\theta}}{1 - \mu_k \tan\theta}\right)' = 0$$

Find the derivative using the quotient rule:

$$\left(\frac{1 + \frac{1}{\tan\theta}\mu_k}{1 - \tan\theta\,\mu_k}\right)' = \frac{\cot^3(\theta) + \cot(\theta)}{(\mu_k - \cot(\theta))^2} = 0$$

The derivative does not have any solutions, but it doesn’t exist at:

$$\mu_k = \cot\theta = \frac{1}{\tan\theta}$$

However, this makes the limit approach infinity, meaning we would have an infinite velocity. In practice, however, the value of μ_k could reach a maximum value of about 1.7, for example. If we assume that this is a sensible estimate and knowing that the derived function is a monotonically increasing function, the actual maximum velocity limit is limited by the maximum value of the coefficient of kinetic friction, and given by the function:

$$v_o \sqrt{\frac{1 + \frac{\mu_k}{\tan\theta}}{1 - \mu_k \tan\theta}}$$

We summarize the results and answer the questions directly:

- Acceleration introduces a linear acceleration term into the equation.
- The net acceleration increases, as now the vector is composed of the linear and the radial acceleration terms.
- The increase in velocity leads to an increase in the radial acceleration term, which must be balanced by a newly introduced friction force due to the tires of the car.
- No change in the magnitude or direction of the normal force and the force of gravity.
- A newly introduced friction force would be pointing along the horizontal x-axis shown on the free-body diagram.
- The net force would deviate from its direction towards the center of the circular motion.
- There would be an upper limit for the velocity based on the equation derived, the maximum velocity is achieved at the maximum value of the value of the kinetic friction.