FIN5DER SEM 2, 2021

Tutorial 9 Questions and Solutions

Problem 15.1.

What does the Black–Scholes–Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the probability distribution of the continuously compounded rate of return on the stock during the year?

The Black–Scholes–Merton option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed.

Problem 15.2.

The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility.

In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\Delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

Problem 15.3.

Explain the principle of risk-neutral valuation.

The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

Problem 15.4.

Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of $50 when the current stock price is $50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$50N(-0.0917)e^{-0.1\times0.25} - 50N(-0.2417)$$

$$= 50 \times 0.4634e^{-0.1\times0.25} - 50 \times 0.4045 = 2.37$$

or $2.37.

Problem 15.5.

What difference does it make to your calculations in Problem 15.4 if a dividend of $1.50 is expected in two months?

In this case we must subtract the present value of the dividend from the stock price before using Black–Scholes-Merton. Hence the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667\times 0.1} = 48.52$$

As before $K = 50$, $r = 0.1$, $\sigma = 0.3$, and $T = 0.25$. In this case

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1\times 0.25} - 48.52N(-0.0414)$$

$$= 50 \times 0.5432e^{-0.1\times 0.25} - 48.52 \times 0.4835 = 3.03$$

or $3.03.

Problem 15.6.

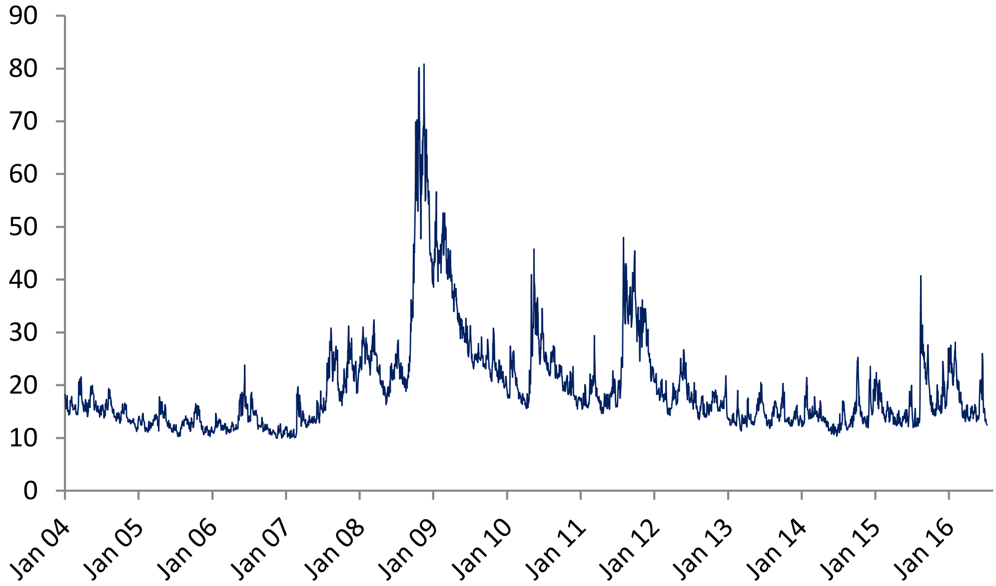
What is implied volatility? How can it be calculated?

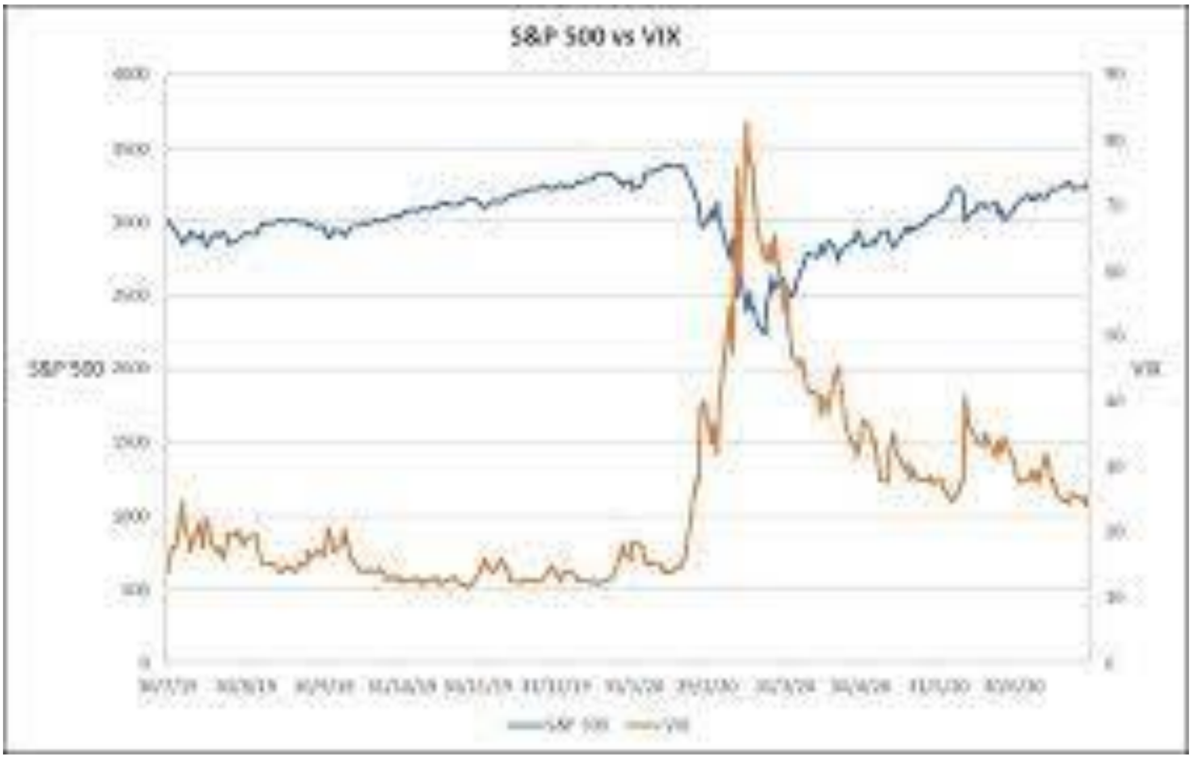
The implied volatility is the volatility that makes the Black–Scholes-Merton price of an option equal to its market price. It is calculated using an iterative procedure.

There is a one-to-one correspondence between prices and implied volatilities

Traders and brokers often quote implied volatilities rather than dollar prices

The VIX S&P500 Volatility Index, or the 'Fear' Index





Problem 15.7.

A stock price is currently $40. Assume that the expected return from the stock is 15% and its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a two-year period?

Note that:

The lognormal property of stock prices can be used to provide information on the probability distribution of the continuously compounded rate of return earned on a stock between times 0 and T. If we define the continuously compounded rate of return per annum realized between times 0 and T as x, then

$$S_T = S_0 e^{xT}$$

so that

$$x = \frac{1}{T} \ln \frac{S_T}{S_0} \qquad \textbf{(15.6)}$$

From equation (15.2), it follows that

$$x \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right) \qquad \textbf{(15.7)}$$

In this case, Problem 15.7, we have a mean and standard deviation of: $\mu = 0.15$ and $\sigma = 0.25$ respectively.

From equation (15.7) the probability distribution for the rate of return over a two-year period with continuous compounding is:

$$\varphi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25^2}{2}\right)$$

i.e.,

$$\varphi(0.11875, 0.03125)$$

The expected value of the return is **11.875%** per annum and the standard deviation is **17.7%** per annum.

Problem 15.13.

What is the price of a European call option on a non-dividend-paying stock when the stock price is $52, the strike price is $50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$

$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$\begin{aligned} &52N(0.5365) - 50e^{-0.12\times0.25}N(0.3865) \\ &= 52\times0.7042 - 50e^{-0.03}\times0.6504 \\ &= 5.06 \end{aligned}$$

or $5.06.

Problem 15.14.

What is the price of a European put option on a non-dividend-paying stock when the stock price is $69, the strike price is $70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$\begin{aligned} & 70e^{-0.05\times 0.5} N(0.0809) - 69N(-0.1666) \\ & = 70e^{-0.025} \times 0.5323 - 69 \times 0.4338 \\ & = 6.40 \end{aligned}$$

or $6.40.

Problem 19.2.

What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

Delta (Δ) is the rate of change of the option price with respect to the price of the underlying asset.

$$\Delta = \frac{\text{Change in option price}}{\text{Change in price of underlying aset}} (= \text{hedge ratio})$$

A delta of 0.7 means that, when the price of the stock increases by a small amount, the price of the option increases by 70% of this amount. Similarly, when the price of the stock decreases by a small amount, the price of the option decreases by 70% of this amount. A short position in 1,000 options has a delta of -700 and can be made delta neutral with the purchase of 700 shares.

Problem 19.3.

Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

Note that:

$$\Delta_c = \frac{f_u - f_d}{S_u - S_d} = \frac{\Delta \mathrm{c}}{\Delta \mathrm{S}} = \frac{\partial c}{\partial S} = N(d_1) > 0$$

In this case, $S_0 = K$, $r = 0.1$, $\sigma = 0.25$, and $T = 0.5$. Also,

$$d_1 = \frac{\ln(S_0 / K) + (0.1 + 0.25^2 / 2)0.5}{0.25\sqrt{0.5}} = 0.3712$$

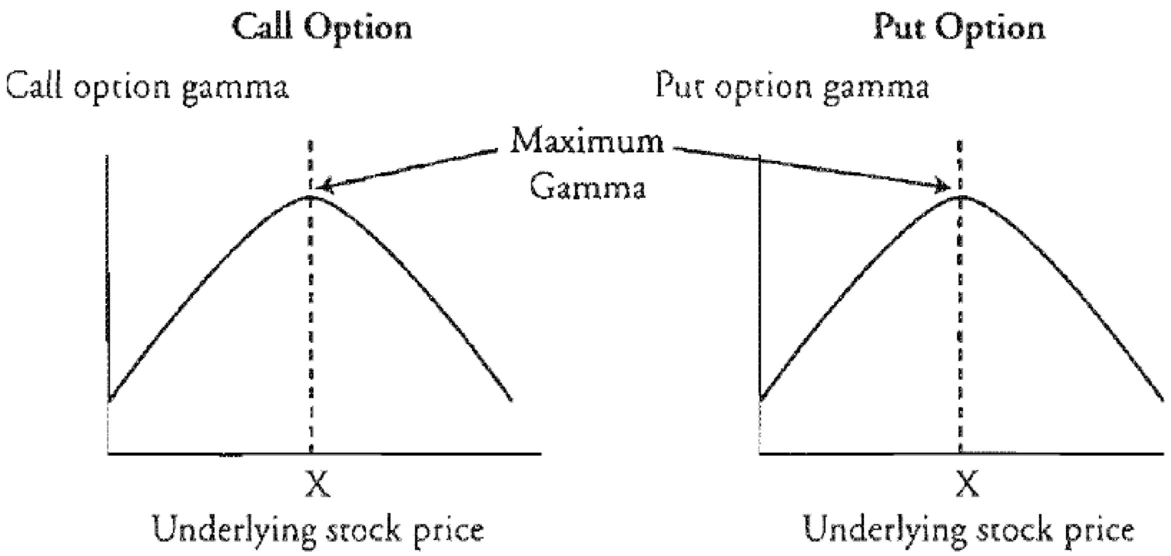
The delta of the option is $N(d_1)$ or 0.64.

Problem 19.5.

What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is large and negative and the delta is zero?

The gamma of an option position is the rate of change of the delta of the position with respect to the asset price.

Gamma measures how well a dynamic hedging will perform when it is not rebalanced in response to stock price changes.



Gamma is largest when the option is at the money. Gamma is close to zero if the option is deep in the money or out of money.

For example, a gamma of 0.1 would indicate that when the asset price increases by a certain small amount delta increases by 0.1 of this amount. When the gamma of an option writer's position is large and negative and the delta is zero, the option writer will lose significant amounts of money if there is a large movement (either an increase or a decrease) in the asset price.

Problem 19.23.

Use the put–call parity relationship to derive, for a non-dividend-paying stock, the relationship between:

(a) *The delta of a European call and the delta of a European put.*
(b) *The gamma of a European call and the gamma of a European put.*
(c) *The vega of a European call and the vega of a European put.*

(a) For a non-dividend paying stock, put-call parity gives at a general time t:

$$p + S = c + Ke^{-r(T-t)}$$

Differentiating with respect to S:

$$\frac{\partial p}{\partial S} + 1 = \frac{\partial c}{\partial S}$$

or

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - 1$$

This shows that the delta of a European put equals the delta of the corresponding European call less 1.0.

(b) Differentiating with respect to S again

$$\frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

Hence the gamma of a European put equals the gamma of a European call.

(c) Differentiating the put-call parity relationship with respect to σ

$$\frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma}$$

showing that the vega of a European put equals the vega of a European call.