3-D Harmonic Potential

Areal velocity

The force is given by:

$$\mathbf{F} = -kr\hat{\mathbf{r}} \tag{1}$$

Since this is a central force, angular momentum is conserved:

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = 0 \tag{2}$$

From the fact that $\mathbf{L}$ is conserved you can deduce that the motion will take place in a plane that contains the point $\mathbf{r} = 0$ which has a normal vector parallel to the angular momentum $\mathbf{L}$. This means that we can treat this problem as a two-dimensional problem.

In systems where angular momentum is conserved the so-called "areal velocity" is constant. This means that the line from the particle to the center sweeps out a constant area per unit time. To see this write the magnitude of the angular momentum as:

$$L = mr^2 \frac{d\theta}{dt} = mrv_\theta \tag{3}$$

Per unit time the distance traveled in the tangential direction is v_θ and the area the line from the center is sweeping out per unit time is rv_θ. From the above equation you see that this must be constant because L is conserved.

Time average of kinetic and potential energies

This particular problem simplifies in Cartesian coordinates. The equation of motion is:

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} \tag{4}$$

If you take the xy plane to be the plane in which the particle moves and take the x and y components of Eq. (4) you see that the problem decouples:

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{k}{m}x \\ \frac{d^2y}{dt^2} &= -\frac{k}{m}y \\ \frac{d^2z}{dt^2} &= -\frac{k}{m}z \end{aligned} \tag{5}$$

The solution is:

$$\begin{aligned} x(t) &= A_1 \sin(\omega t + \phi_1) \\ y(t) &= A_2 \sin(\omega t + \phi_2) \\ z(t) &= 0 \end{aligned} \tag{6}$$

where

$$\omega = \sqrt{\frac{k}{m}} \tag{7}$$

is the angular frequency.

The potential that corresponds to the force (1) is:

$$V(r) = \frac{k}{2} r^2 = \frac{k}{2}\left(x^2 + y^2 + z^2\right) \tag{8}$$

Note that the force is given by minus the gradient of the potential. If we insert the solution of the equation of motion (6) in the potential we obtain:

$$V = \frac{k}{2}\left[A_1^2 \sin^2(\omega t + \phi_1) + A_2^2 \sin^2(\omega t + \phi_2)\right] \tag{9}$$

The average over one period is:

$$\langle V \rangle = \frac{k}{4}\left(A_1^2 + A_2^2\right) \tag{10}$$

Here we have used that the average of $sin^2(\omega t + \phi)$ over one period is $\frac{1}{2}$

The kinetic energy is given by:

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2} m \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \\ &= \frac{k}{2}\left[A_1^2 \cos^2(\omega t + \phi_1) + A_2^2 \cos^2(\omega t + \phi_2)\right] \end{aligned} \tag{11}$$

here we have used (7) to eliminate ω in favor of k. By adding the expressions for E_{kin} and V given in Eq. (11) and Eq. (9) respectively, you obtain the total energy:

$$E = E_{\text{kin}} + V = \frac{k}{2}\left(A_1^2 + A_2^2\right) \tag{12}$$

Here we have used that $\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi) = 1$ You thus see that the average of the potential energy (10) is half of the total energy, so the average of the kinetic energy is equal to the average of the potential energy. This is to be expected from the virial theorem. The viral theorem states that:

$$2\left\langle E_{\text{kin}} \right\rangle = n \left\langle V \right\rangle \tag{13}$$

where n is the power of r in the potential energy term describing the particle interaction. In this case $n = 2$, so the viral theorem yields the same result.

Derivation without using equation of motion

It is also possible to calculate the average of the potential energy without solving for the equation of motion. This is useful in cases where no simple solution is available. This works as follows. The total energy in polar coordinates in the plane of motion:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{1}{2}kr^2 \tag{14}$$

And you find that the radial velocity is given by

$$\dot{r} = \pm\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} + \frac{1}{2}kr^2\right)} \tag{15}$$

Consider the time average of some function $f(r)$ over the time interval from $t = 0$ to T. We can write it as:

$$\langle f\rangle = \frac{1}{T}\int_0^T f(r(t))\,dt = \frac{1}{T}\int_{r_1}^{r_2} f(r)\,\frac{dr}{\dot{r}} \tag{16}$$

In the previous section we averaged over one period, but it is enough to average over half a period. This is the time needed to move from one turning point to the other turning point. At the turning points $\dot{r} = 0$. You can assume that the particle is moving toward larger r and take the plus sign in (15). The time T in the above equation can then be expressed as:

$$T = \int_0^T dt = \int_{r_1}^{r_2} \frac{dr}{\dot{r}} = \sqrt{\frac{m}{2}}\int_{r_1}^{r_2} \frac{dr}{\sqrt{E - \frac{L^2}{2mr^2} + \frac{1}{2}kr^2}} \tag{17}$$

Here r_1 and r_2 are the two positive zeros of $\dot{r}$. You can calculate this integral by substituting $r = \sqrt{x}$:

$$T = \frac{\pi}{2}\sqrt{\frac{m}{k}\frac{(x_2 - x_1)}{2}} \tag{18}$$

Here $x_i = r_i^2$ and they correspond to the zeroes of the denominator in the integrand:

$$x_{1,2} = \frac{E}{k} \pm \sqrt{\left(\frac{E}{k}\right)^2 - \frac{L^2}{mk}} \tag{19}$$

If you now insert the expression for the kinetic energy for the function in (16) and calculate the integral in the same way as the integral in Eq. (17) for T you find that the average is $\frac{E}{2}$, in agreement with the result of the previous section.