(2) a) $F_n^2 - F_{n+1}F_{n-1} = (-1)^n$ for $n \geq 1$.

We will prove by induction on n.

1) Base case $p(1): F_1^2 - F_2F_0 = (-1)^1$

$F(0) = 1 \quad F(1) = 1 \quad F(2) = 1+1 = 2$

$1^2 - 2 \cdot 1 = -1 = (-1)^1 = -1$

2) Let $p(k): F_k^2 - F_{k+1}F_{k-1} = (-1)^k$ holds for $k > 1$.

3) We need to show

$p(k+1): F_{k+1}^2 - F_{k+2} \cdot F_k = (-1)^{k+1}$

$F_{k+2} \cdot F_k = (F_k + F_{k+1})F_k = F_k^2 + F_k F_{k+1} =$

induction

$= (F_{k+1} \cdot F_{k-1} + (-1)^k) + F_k F_{k+1} =$

$= F_{k+1} \cdot (F_k + F_{k-1}) + (-1)^k =$

$= F_{k+1} \cdot F_{k+1} + (-1)^k =$

$= F_{k+1}^2 - (-1)^{k+1}$

b) $\sum_{i=0}^{n} F_i = F_{n+2} - 1$

We show by induction on n.

1) Base case $n=0$

We know that $F_0 = F_1 = 1$, $F_2 = 2$.

$F_0 = F_2 - 1 \iff 1 = 2 - 1 \iff 1 = 1$

2) Let $\sum_{i=0}^{k} F_k = F_{k+2} - 1$ for $k \geq 1$

induction 2)

3) $\sum_{i=0}^{k+1} F_k = \sum_{i=0}^{k} F_k + F_{k+1} = (F_{k+2} - 1) + F_{k+1} =$

$= (F_{k+1} + F_{k+2}) - 1 = F_{k+3} - 1$

c) $F_{n-1}^2 + F_n^2 = F_{2n}$

$F_{n-1}F_n + F_nF_{n+1} = F_{2n+1}$

We prove by induction on n

1) For $n=1$ we have

$F_0^2 + F_1^2 = F_2$

$F_0F_1 + F_1F_2 = F_3$

$F_0 = F_1 = 1$, $F_2 = 2$, $F_3 = 3$

$F_0^2 + F_1^2 = 1^2 + 1^2 = 2 = F_2$

$F_0F_1 + F_1F_2 = 1 \cdot 1 + 1 \cdot 2 = 3 = F_3$.

2) For $k>1$ we suppose that

$$F_{k-1}^2 + F_k^2 = F_{2k}$$

$$F_{k-1}F_k + F_k F_{k+1} = F_{2k+1}$$

3) $F_{2k+2} = F_{2k+1} + F_{2k} =$

$$= F_{k-1}F_k + F_k F_{k+1} + F_{k-1}^2 + F_k^2 =$$

$$= F_{k-1}(F_k + F_{k-1}) + F_k F_{k+1} + F_k^2 =$$

$$= F_{k-1} \cdot F_{k+1} + F_k F_{k+1} + F_k^2 =$$

$$= F_{k+1}(F_{k-1} + F_k) + F_k^2 = F_{k+1} \cdot F_{k+1} + F_k^2 =$$

$$= F_{k+1}^2 + F_k^2 .$$

$$F_{2k+3} = F_{2k+2} + F_{2k+1} = \underline{F_{k+1}^2} + F_k^2 + F_{k-1}F_k + \underline{F_k F_{k+1}}$$

$$= F_{k+1}(F_{k+1} + F_k) + F_k(F_k + F_{k-1}) =$$

$$= F_{k+1} \cdot F_{k+2} + F_k \cdot F_{k+1}$$

Scanned with CamScanner

d) $F_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}$

F_n - the number of representations for n as an ordered sum of 1s and 2s.

If we have i-number of 2s we have $n-2i$ 1s, thus, $n-i$ summands.

Therefore, there are $\binom{n-i}{i}$ ways to choose the positions of the 2s in the sum.

Thus

$$F_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}.$$

③ F_n - composite for all odd $n \geq 3$.

From the question 2.(c) we know that

$$F_{2k+1} = F_{k-1}F_k + F_k F_{k+1} =$$

$$= F_k \cdot (F_{k-1} + F_{k+1})$$

F_{2k+1} is a product of two positive integers, thus, F_{2k+1} is composite.

Since $n = 2k+1$ is an odd number, we obtain that F_n is composite for all odd $n > 3$.

⑨ a)

i) $f(n+1) = f(n)^2$, $f(0) = 2$

$f(1) = f(0)^2 = 2^2 = 4$

$f(2) = f(1)^2 = 4^2 = 16$

By induction we will show that $f(n) = 2^{2^n}$, for $n = 0, 1$.

1) Base case $n = 0$.

$f(0) = 2$; $f(0) = 2^{2^0} = 2^1 = 2$

2) Let $f(n) = 2^{2^n}$ for $n > 0$.

3) $f(n+1) = f(n)^2 = (2^{2^n})^2 = 2^{2^{n+1}}$

Thus $f(n) = 2^{2^n}$

Scanned with CamScanner

ii) $f(n+1)=f(n)+f(n-1)+f(n-2)$

$$f(0)=f(1)=f(2)=1$$

The characteristic equation for the given relation is

$$x^3-x^2-x-1=0.$$

Let α, β, γ are the roots of the equation $x^3-x^2-x-1=0$

Thus $f(n)=a\alpha^n+b\beta^n+c\gamma^n$

$$f(0)=a\alpha^0+b\beta^0+c\gamma^0=a+b+c=1$$

$$f(1)=a\alpha+b\beta+c\gamma=1$$

$$f(2)=a\alpha^2+b\beta^2+c\gamma^2=1$$

Solving the system

$$\begin{cases} a+b+c=1 \\ a\alpha+b\beta+c\gamma=1 \\ a\alpha^2+b\beta^2+c\gamma^2=1 \end{cases}$$

where a, b, c are unknowns we find a, b and c.

iii) $f(n+1) = 1 + \sum_{i=0}^{n-1} f(i)$, $f(0) = 1$

$$f(n+1) = 1 + \sum_{i=0}^{n-2} f(i) + f(n-1) =$$

$$= 1 + f(n-1) + \sum_{i=0}^{n-2} f(i) =$$

$$= 1 + f(n-1) + f(n) - 1 =$$

$$= f(n) + f(n-1).$$

$n=0$; $f(1) = 1 +$ empty sum

$$f(1) = 1$$

$$f(2) = f(1) + f(0) = 1 + 1 = 2$$

$$f(3) = f(2) + f(1) = 2 + 1 = 3$$

$$f(4) = f(3) + f(2) = 3 + 2 = 5$$

$1, 1, 2, 3, 5, \ldots$

$\{f(n)\}_{n=1}^{\infty}$ is Fibonacci sequence.

6) $f(n)$ - the number of ways of writing n as a sum of positive integers

1 summand: $n = n$ - 1 way

if n is represented as a sum of at least 2 summands we have

$n = i + (n-i)$ $\quad 1 \leq i \leq n-1$

- where represented on $f(n-1)$ ways

Thus
$$f(n) = 1 + \sum_{i=1}^{n-1} f(n-i) =$$

$$= 1 + \sum_{i=1}^{n-1} f(i) = 1 + f(n-1) + \sum_{i=1}^{n-2} f(i)$$

$$= 1 + f(n-1) + (f(n-1) - 1)$$

$$f(n) = 2f(n-1) = 2 \cdot 2f(n-2) = 2^2 f(n-2) =$$

$$= \ldots = 2^{n-1} f(1) = 2^{n-1}$$

(15) a) Let $f(n)$ denote the number of ways of counting.

We know that A leads by just one vote offer the first vote is counted.

Let us suppose that this next holds after $2i+1$ votes have been counted.

Since A is first and both of the candidate receives i votes, we obtain that there are $f(i)$ choices for the count between these points. Therefore for the rest of the count there exist $f(n-i)$ choices.

Thus, we have

$$f(n) = f(1) \cdot f(n-1) + f(2) \cdot f(n-2) + \dots + f(n-1) f(1)$$

$$= \sum_{i=1}^{n-1} f(i) f(n-i) \quad \text{Catolan recurrence.}$$

$$\parallel$$

$$C_n$$

6) According to the new election rules we know that A receives an extra vote at the start and B receives an extra vote at the end, thus, we have $n+1$ votes For A and $n+1$ votes for B, and A is always ahead.

Therefore, there are C_{n+1} different ways of counting (From (a)).

(16) Since there are $2n$ balls, there are $\binom{2n}{n}$ ways in which the balls could be drawn.

If the number of red balls never exceeds the number of blue balls, then the clown stays dry. The number of ways of counting is equal to C_{n+1}, by using question 15.

We know that $C_{n+1} = \frac{1}{n+1}\binom{2n}{n}$

where C_{n+1} is the $(n+1)$th Catalan number.

The probability that the clown remains dry is
$$\frac{C_{n+1}}{\binom{2n}{n}} = \frac{\frac{1}{n+1}\cancel{\binom{2n}{n}}}{\cancel{\binom{2n}{n}}} = \frac{1}{n+1}$$